Fredholm representations of solutions to the KPI equation, their wronkian versions and rogue waves

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Abstract

We construct solutions to the Kadomtsev-Petviashvili equation (KPI) in terms of Fredholm determinants. We deduce solutions written as a quotient of wronskians of order $2N$. These solutions called solutions of order N depend on $2N-1$ parameters. When one of these parameters tends to zero, we obtain N order rational solutions expressed as a quotient of two polynomials of degree $2N(N+1)$ in x, y and t depending on $2N-2$ parameters.
So we get with this method an infinite hierarchy of solutions to the KPI equation.

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1 Introduction

The Kadomtsev-Petviashvili equation is a well-known nonlinear partial differential equation in two spatial and one temporal coordinates. There are two distinct versions of the KP equation, which can be written in the form :

$$(4u_t - 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0. \tag{1}$$

As usual, subscripts x, y and t denote partial derivatives, and $\sigma^2 = \pm 1$. The case $\sigma = 1$ is known as the KPII equation, and the case $\sigma = i$ as the KPI equation.

The KP equation is a universal integrable system in two spatial dimensions and has been extensively studied in the mathematical community.
The two versions of the KP equation significantly differ with respect to their underlying mathematical structure and the behavior of their solutions, despite their apparent similarity.
The KP equation first appeared in 1970, in a paper written by Kadomtsev and Petviashvili [41]. For example, this equation is considered as a model for surface and internal water waves by Ablowitz and Segur [1], and in nonlinear optics by Pelinovsky, Stepanyants and Kivshar [57]. The discovery of the KP equation happened almost simultaneously with the development of the inverse scattering transform (IST) as it is explained in Manakov et al. [51]. This method to construct solution of the initial-value problem for nonlinear partial differential equations was originally developed for equations in one spatial dimension. However, in 1974 Dryuma showed how the KP equation could be written in Lax form [8]. Then, Zakharov extended the IST to equations in two spatial dimensions, including the KP equation, and obtained several exact solutions to the KP equation.
It was Dubrovin who constructed for the first time in 1981 [9] the solutions to KPI given in terms of Riemann theta functions in the frame of algebraic geometry .
From the 1980's, a lot of methods have been carried out to solve that equation, like for example the nonlocal Riemann-Hilbert problem, the d-bar problem or inverse scattering problem using integration in the complex plane as it is reviewed in the book by Ablowitz and Clarkson published in 1991 [2]. There is a wealth of papers which deal with solutions to the KPI equation. We can cite in particular the works of Krichever [48], Satsuma [58], Matveev [**?**], Veselov [59], Freeman [10], Weiss [60], Latham [49], Pelinovski [55, 56], Boiti [7], Ablowitz [3], Biondini [4], Kodama ,[47], Matveev [**?**], Ma [50].

The paper is organized as follows. First of all, we express the solutions in terms of Fredholm determinants of order $2N$ depending on $2N-1$ parameters. We deduce another representation in term of wronskians of order $2N$ with $2N-1$ parameters. This representation allows to obtain an infinite hierarchy of solutions to the KPI equation, depending on $2N-1$ real parameters .
Then we use these results to construct rational solutions to the KPI equation. These results represent a new method to build multi rogue waves. New rational solutions depending a priori on $2N-2$ parameters at order N are constructed, when one parameter tends towards 0.
Families depending on $2N-2$ parameters for the N-th order as a ratio of two polynomials of x, y and t of degree $2N(N+1)$ are obtained.
That provides an effective method to construct an infinite hierarchy of rational solutions of order N dependent on $2N-2$ real parameters. We present here only the rational solutions of order 3, dependent on 4 real parameters, and the representations of their modulus in the plane of the coordinates (x,y) according to the real parameters a_1, b_1, a_2, b_2 and time t.

2 Expression of solutions to KPI equation in terms of Fredholm determinants

In the following, we need to define some notations. First of all, we define real numbers λ_j such that $-1 < \lambda_\nu < 1$, $\nu = 1, \dots, 2N$ which depend on a parameter ϵ which will be intended to tend towards 0; they can be written as

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \le j \le N, \tag{2}$$

The terms $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ are functions of λ_ν, $1 \le \nu \le 2N$; they are defined by the formulas :

$$\begin{array}{l} \kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}};\\ x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad r = 1,3, \quad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2},\\ \kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1},\\ x_{r,N+j} = -x_{r,j},, \quad \tau_{N+j} = \tau_j \quad j = 1, \dots, N. \end{array} \tag{3}$$

e_ν $1 \le \nu \le 2N$ are defined in the following way :

$$\begin{array}{l} e_j = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} - i\sum_{k=1}^{1/2\,M-1} b_K (je)^{2\,k+1}\right),\\ e_{N+j} = 2i\left(\sum_{k=1}^{1/2\,M-1} a_k (je)^{2\,k+1} + i\sum_{k=1}^{1/2\,M-1} b_k (je)^{2\,k+1}\right), \quad 1 \le j \le N,\\ a_k, b_k \in \mathbf{R}, \quad 1 \le k \le N. \end{array} \tag{4}$$

ϵ_ν, $1 \le \nu \le 2N$ are real numbers defined by :

$$e_j = 1, \quad e_{N+j} = 0 \quad 1 \le j \le N. \tag{5}$$

Let I be the unit matrix and $D_r = (d_{jk})_{1 \le j,k \le 2N}$ the matrix defined by :

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \ne \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu}\right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{6}$$

Then we have the following result :

Theorem 2.1 *The function v defined by*

$$v(x,y,t) = -2\,\frac{|n(x,y,t)|^2}{d(x,y,t)^2} \tag{7}$$

where

$$n(x,y,t) = \det(I + D_3(x,y,t)), \tag{8}$$

$$d(x,y,t) = \det(I + D_1(x,y,t)), \tag{9}$$

and $D_r = (d_{jk})_{1 \leq j,k \leq 2N}$ *the matrix*

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{10}$$

is a solution to the KPI equation (1), dependent on $2N-1$ *parameters* a_k, b_h, $1 \leq k \leq N-1$ *and* ϵ.

3 Expression of solutions to the KPI equation in terms of wronskians

We want to express solutions to the NLS equation in terms of wronskians. Thus, we need the following notations :

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \leq \nu \leq N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \leq \nu \leq 2N, \quad r = 1,3, \tag{11}$$

with the arguments

$$\Theta_{r,\nu} = \tfrac{\kappa_\nu x}{2} + i\delta_\nu y - i\tfrac{x_{r,\nu}}{2} - i\tfrac{\tau_\nu}{2} t + \gamma_\nu w - i\tfrac{e_\nu}{2}, \quad 1 \leq \nu \leq 2N. \tag{12}$$

We denote $W_r(w)$ the wronskian of the functions $\phi_{r,1}, \ldots, \phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1} \phi_{r,\nu})_{\nu,\, \mu \in [1,\ldots,2N]}]. \tag{13}$$

We consider the matrix $D_r = (d_{\nu\mu})_{\nu,\, \mu \in [1,\ldots,2N]}$ defined in (10). Then we have the following statement

Theorem 3.1

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \ldots, \phi_{r,2N})(0), \tag{14}$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

From the initial formulation, the solution v to the KPI equation can be written as

$$v(x,y,t) = -2 \frac{|\det(I + D_3(x,y,t))|^2}{(\det(I + D_1(x,y,t)))^2}.$$

Using (14), the following relation between Fredholm determinants and wronskians is obtained

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \ldots, \phi_{r,2N})(0)$$

and

$$\det(I + D_1) = k_1(0) \times W_1(\phi_{r,1}, \ldots, \phi_{r,2N})(0).$$

As $\Theta_{3,j}(0)$ contains N terms $x_{3,j}$ $1 \leq j \leq N$ and N terms $-x_{3,j}$ $1 \leq j \leq N$, we have the equality $k_3(0) = k_1(0)$, and we get the following result :

Theorem 3.2 *The function v defined by*

$$v(x,y,t) = -2\frac{|W_3(\phi_{3,1},\dots,\phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1},\dots,\phi_{1,2N})(0))^2}$$

is a solution KPI equation depending on $2N-1$ real parameters a_k, b_k and ϵ, with ϕ^r_ν defined in (11)

$$\begin{array}{ll} \phi_{r,\nu} = \sin(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}), & 1 \leq \nu \leq N, \\ \phi_{r,\nu} = \cos(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}), & N+1 \leq \nu \leq 2N, \quad r = 1,3, \end{array}$$

κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν being defined in(3), (2) and (4).

4 The limit case when ϵ tends to 0

4.1 Families of rational solutions of order N depending on $2N-2$ parameters

We obtain here families of rational solutions to the KPI equation depending on $2N-2$ parameters. To get it, it is sufficient to make the parameter ϵ tend to 0. Then, we get the following result :

Theorem 4.1 *The function v defined by*

$$v(x,y,t) = \lim_{\epsilon \to 0} -2\frac{|W_3(x,y,t)|^2}{(W_3(x,y,t))^2} \tag{15}$$

is a rational solution to the KPI equation (1) quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters $\tilde{a_j}$ and $\tilde{b_j}$, $1 \leq j \leq N-1$ of degrees $2N(N+1)$ in x, y and t.

4.2 Rational solutions of order 3 depending on 4 parameters

In the following, we explicitly construct rational solutions to the KPI equation of order 3 depending on 4 parameters.
Because of the length of the expression, we cannot give it in this paper. We only give the expression without parameters and we present it in the appendix.

We give patterns of the modulus of the solutions in the plane (x,y) of coordinates in functions of parameters a_1, b_1, a_2, b_2 and time t.
The solutions of KP equation being derived from the solutions of the NLS equation, one recovers similarity with figures already obtained by Akhmediev et al. [45, 44, 46] and the author [23, 35] in the case of the NLS equation.

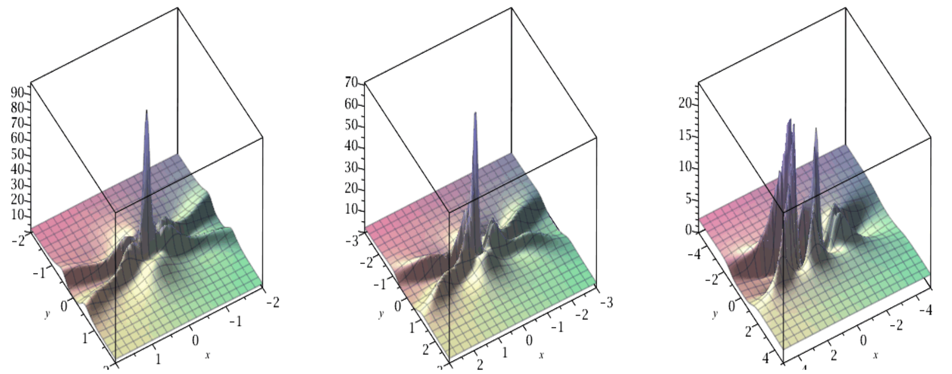

Figure 1. Solution of order 3 to KPI, on the left for $t = 0$; in the center for $t = 0,01$; on the right for $t = 0,1$; all the parameters are equal to 0.

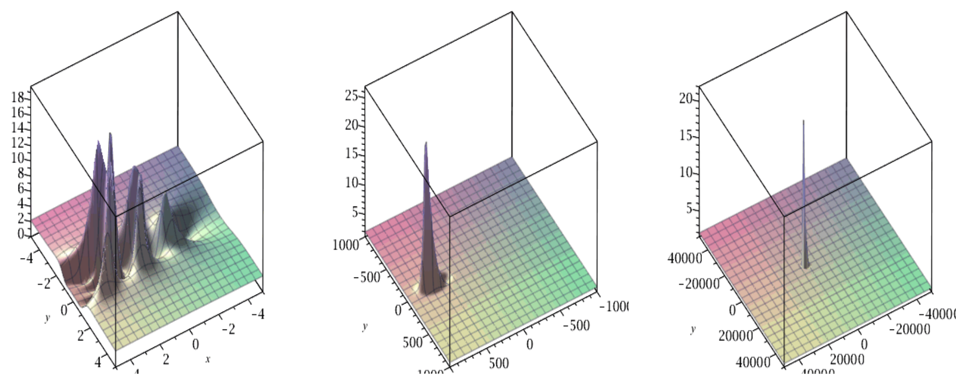

Figure 2. Solution of order 3 to KPI, on the left for $t = 0,2$; in the center for $t = 10^2$; on the right for $t = 10^3$; all the parameters are equal to 0.

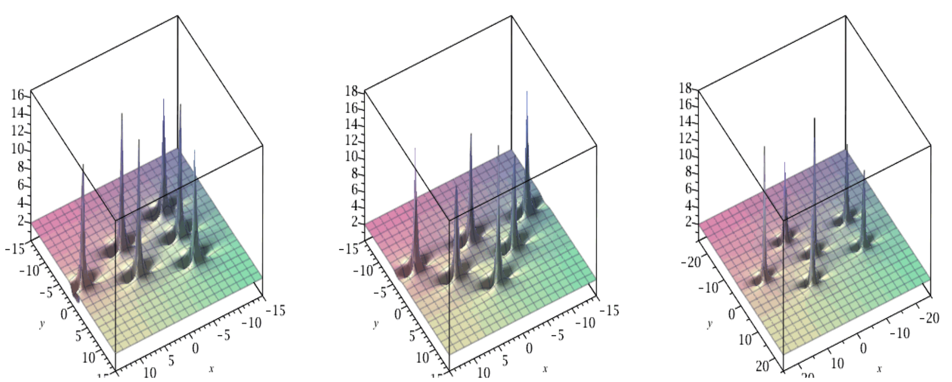

Figure 3. Solution of order 3 to KPI, on the left for $a_1 = 10^3$; in the center for $b_1 = 10^3$; on the right for $a_2 = 10^6$; here $t = 0$.

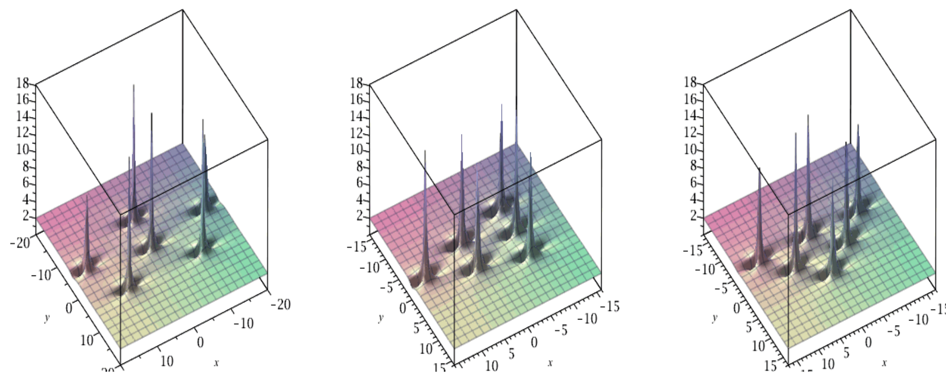

Figure 4. Solution of order 3 to KPI, on the left for $t = 0$, $b_2 = 10^6$; in the center for $t = 0,01$, $a_1 = 10^3$ all the other parameters are equal to 0; on the right for $t = 0,1$, $b_1 = 10^3$ all the parameters are equal to 0.

5 Conclusion

In this article, solutions to the KPI equation have been built, starting from the solutions of the nonlinear Schrödinger equation in terms of wronskians and what made it possible to obtain rational solutions in terms of quotients of two polynomials of degree $2N(N+1)$ in x, y and t depending on $2N-2$ parameters. Other approaches to build solutions of KPI equation in terms of wronskians have been led and ones can be mentionned those most significant. In 1989, solutions were built in relation to time dependent Schrödinger equation [6]. The expressions of the solutions given were however not explicit. The following year, Hirota and Ohta [40] built, the solutions as particular case of a hierarchy of coupled bilinear equations given in terms of Pfaffians. In 1993, Oevel [53] used the Darboux transformations to obtain among others the solutions of the multicomponent KP hierarchy. No explicit solutions were also given. In the following article published in 1996 [54], the same author gave explicit solutions in terms of wronskians of order 2 but different from the ones we have constructed in this paper. More recently, in 2013 [43], wronskians identities of bilinear KP hierarchy were given which has made it possible to search new wronskian solutions of PDE's. In 2014, using iterated Darboux transformations, in particular, solutions of KPI equation were constructed in terms of reduced multicomponent wronskian solutions [61]. In the later study, an explicit solution at order 1 was built but different from the one which we have given in this article. Only one asymptotic study has been carried out for order higher than 2.
Here we have given a new method to construct solutions to the KPI equation. We have constructed two types of representations of the solutions to the KPI equation at order N. We have given an expression in terms of Fredholm determinants of order $2N$ depending on $2N-1$ real parameters. We have also given a representation in terms of wronskians of order $2N$ depending on $2N-1$

real parameters. When one of parameters (ϵ) tends to zero, we obtain rational solutions to the KPI equation depending on $2N-2$ real parameters. In a forthcoming paper, we will give a general formulation of rational solutions to the KPI without limit at order N which will depend on $2N-2$ real parameters. We will show that these solutions can be expressed in terms of polynomials of degree $2N(N+1)$ in x, y and t; we will prove that the maximum of the modulus of these solutions is equal to $2(2N+1)^2$. We will give a systematic approach to find explicit solutions for higher orders and try to describe the structure of these rational solutions.

Acknowledgements
I would like to thank V.B. Matveev for having asked me the question about the links between the solutions to NLS equation which I had built and those to KPI equation, as well as its comments concerning the history of the KP equation.
I would like also to thank B. Dubrovin for the discussions we had in Gallipoli which made me understand that it was possible to construct solutions of KP from those of NLS.

Appendix The solutions to the KPI equation which we have constructed can be written as

$$v_3(x,y,t) = -2\frac{|n_3(x,y,t)|^2}{(d_3(x,y,t))^2}$$

with

$$\frac{n_3(x,y,t)}{d_3(x,y,t)} = \frac{F_3(2x,4y,4t) - iH_3(2x,4y,4t)}{Q_3(2x,4y,4t)}$$

with

$$\begin{array}{l} F_3(X,Y,T) = \sum_{k=0}^{12} f_k(Y,T)X^k, \\ H_3(X,Y,T) = \sum_{k=0}^{12} h_k(Y,T)X^k, \\ Q_3(X,Y,T) = \sum_{k=0}^{12} q_k(Y,T)X^k. \end{array}$$

$\mathbf{f}_{12} = 1$, $\mathbf{f}_{11} = -36\,T$, $\mathbf{f}_{10} = 594\,T^2+6\,Y^2-18$, $\mathbf{f}_9 = -5940\,T^3+\left(-180\,Y^2+780\right)T$,

$\mathbf{f}_8 = 40095\,T^4+15\,Y^4+\left(2430\,Y^2-13770\right)T^2-450\,Y^2-225$, $\mathbf{f}_7 = -192456\,T^5+(-19440\,Y^2$ $+136080)T^3+\left(-360\,Y^4+10800\,Y^2+5400\right)T$, $\mathbf{f}_6 = 673596\,T^6+20\,Y^6+(102060\,Y^2$ $-850500)T^4-1380\,Y^4+\left(3780\,Y^4-113400\,Y^2-48060\right)T^2+1980\,Y^2-2700$

$\mathbf{f}_5 = -1732104\,T^7+\left(-367416\,Y^2+3551688\right)T^5+\left(-22680\,Y^4+680400\,Y^2+184680\right)T^3$ $+\left(-360\,Y^6+23400\,Y^4+7560\,Y^2+70200\right)T$, $\mathbf{f}_4 = 3247695\,T^8+15\,Y^8+(918540\,Y^2$ $-10103940)T^6-1620\,Y^6+\left(85050\,Y^4-2551500\,Y^2-109350\right)T^4+2250\,Y^4+(2700\,Y^6$ $-164700\,Y^4-251100\,Y^2-1077300)T^2+2700\,Y^2+14175$, $\mathbf{f}_3 = -4330260\,T^9$

$$+\left(-1574640\,Y^2+19420560\right)T^7+\left(-204120\,Y^4+6123600\,Y^2-1603800\right)T^5$$
$$+\left(-10800\,Y^6+615600\,Y^4+1263600\,Y^2+7376400\right)T^3+(-180\,Y^8+17520\,Y^6$$
$$+1800\,Y^4+399600\,Y^2-429300)T,\quad \mathbf{f}_2=3897234\,T^{10}+6\,Y^{10}+(1771470\,Y^2$$
$$-24210090)T^8-810\,Y^8+\left(306180\,Y^4-9185400\,Y^2+5904900\right)T^6+3420\,Y^6$$
$$+\left(24300\,Y^6-1287900\,Y^4-2259900\,Y^2-23886900\right)T^4-83700\,Y^4+(810\,Y^8$$
$$-70200\,Y^6-180900\,Y^4-2446200\,Y^2+3746250)T^2+287550\,Y^2+28350,$$
$$\mathbf{f}_1=-2125764\,T^{11}+\left(-1180980\,Y^2+17714700\right)T^9+(-262440\,Y^4+7873200\,Y^2$$
$$-8660520)T^7+\left(-29160\,Y^6+1428840\,Y^4+612360\,Y^2+36012600\right)T^5$$
$$+\left(-1620\,Y^8+123120\,Y^6+793800\,Y^4+5670000\,Y^2-18638100\right)T^3$$
$$+\left(-36\,Y^{10}+4140\,Y^8+57240\,Y^6+394200\,Y^4-1854900\,Y^2-850500\right)T,$$
$$\mathbf{f}_0=531441\,T^{12}+Y^{12}+\left(354294\,Y^2-5786802\right)T^{10}-138\,Y^{10}+(98415\,Y^4-2952450\,Y^2$$
$$+4822335)T^8-8145\,Y^8+\left(14580\,Y^6-656100\,Y^4+1443420\,Y^2-20047500\right)T^6$$
$$-37260\,Y^6+\left(1215\,Y^8-79380\,Y^6-984150\,Y^4-6002100\,Y^2+40935375\right)T^4+327375\,Y^4$$
$$+\left(54\,Y^{10}-5130\,Y^8-182340\,Y^6-1509300\,Y^4+3106350\,Y^2+1129950\right)T^2+141750\,Y^2-14175$$

$$\mathbf{h}_{12}=0,\quad \mathbf{h}_{11}=0,\quad \mathbf{h}_{10}=24\,Y,\quad \mathbf{h}_9=-720\,TY,\quad \mathbf{h}_8=9720\,T^2Y+120\,Y^3-360\,Y,$$
$$\mathbf{h}_7=-77760\,T^3Y+\left(-2880\,Y^3+8640\,Y\right)T,\quad \mathbf{h}_6=408240\,T^4Y+240\,Y^5-3360\,Y^3$$
$$+\left(30240\,Y^3-90720\,Y\right)T^2-3600\,Y,\quad \mathbf{h}_5=-1469664\,T^5Y+\left(-181440\,Y^3+544320\,Y\right)T^3$$
$$+\left(-4320\,Y^5+48960\,Y^3+99360\,Y\right)T,\quad \mathbf{h}_4=3674160\,T^6Y+240\,Y^7-5040\,Y^5+(680400\,Y^3$$
$$-2041200\,Y)T^4-10800\,Y^3+\left(32400\,Y^5-280800\,Y^3-486000\,Y\right)T^2-32400\,Y$$
$$\mathbf{h}_3=-6298560\,T^7Y+\left(-1632960\,Y^3+4898880\,Y\right)T^5+(-129600\,Y^5+777600\,Y^3$$
$$-1166400\,Y)T^3+\left(-2880\,Y^7+37440\,Y^5-100800\,Y^3+734400\,Y\right)T$$
$$\mathbf{h}_2=7085880\,T^8Y+120\,Y^9-1440\,Y^7+\left(2449440\,Y^3-7348320\,Y\right)T^6+41040\,Y^5$$
$$+\left(291600\,Y^5-972000\,Y^3+14288400\,Y\right)T^4-151200\,Y^3+(12960\,Y^7-64800\,Y^5+1144800\,Y^3$$
$$-3823200\,Y)T^2+113400\,Y,\quad \mathbf{h}_1=-4723920\,T^9Y+\left(-2099520\,Y^3+6298560\,Y\right)T^7$$
$$+\left(-349920\,Y^5+233280\,Y^3-36741600\,Y\right)T^5+(-25920\,Y^7-77760\,Y^5-2980800\,Y^3$$
$$+14904000\,Y)T^3+\left(-720\,Y^9-2880\,Y^7-4320\,Y^5+1425600\,Y^3+874800\,Y\right)T$$
$$\mathbf{h}_0=1417176\,T^{10}Y+24\,Y^{11}+600\,Y^9+\left(787320\,Y^3-2361960\,Y\right)T^8-20880\,Y^7$$
$$+\left(174960\,Y^5+349920\,Y^3+30967920\,Y\right)T^6-231120\,Y^5+(19440\,Y^7+213840\,Y^5$$

$$+2235600\,Y^3-27507600\,Y)T^4-59400\,Y^3+(1080\,Y^9+21600\,Y^7-114480\,Y^5-4644000\,Y^3 -7273800\,Y)T^2+113400\,Y$$

$\mathbf{q}_{12}=1$, $\quad \mathbf{q}_{11}=-36\,T$, $\quad \mathbf{q}_{10}=594\,T^2+6\,Y^2+6$, $\quad \mathbf{q}_9=-5940\,T^3+\left(-180\,Y^2+60\right)T$, $\mathbf{q}_8=40095\,T^4+15\,Y^4+\left(2430\,Y^2-4050\right)T^2-90\,Y^2+135$, $\quad \mathbf{q}_7=-192456\,T^5+(-19440\,Y^2+58320)T^3+\left(-360\,Y^4+2160\,Y^2-3240\right)T$, $\quad \mathbf{q}_6=673596\,T^6+20\,Y^6+(102060\,Y^2-442260)T^4-180\,Y^4+\left(3780\,Y^4-22680\,Y^2+42660\right)T^2+540\,Y^2+2340$
$\mathbf{q}_5=-1732104\,T^7+\left(-367416\,Y^2+2082024\right)T^5+\left(-22680\,Y^4+136080\,Y^2-359640\right)T^3+\left(-360\,Y^6+1800\,Y^4-1080\,Y^2-55080\right)T$, $\quad \mathbf{q}_4=3247695\,T^8+15\,Y^8+(918540\,Y^2-6429780)T^6+60\,Y^6+\left(85050\,Y^4-510300\,Y^2+1931850\right)T^4-1350\,Y^4+(2700\,Y^6-2700\,Y^4+72900\,Y^2+639900)T^2+13500\,Y^2+3375$, $\quad \mathbf{q}_3=-4330260\,T^9+(-1574640\,Y^2+13122000)T^7+\left(-204120\,Y^4+1224720\,Y^2-6502680\right)T^5+(-10800\,Y^6-32400\,Y^4-1069200\,Y^2-4676400)T^3+\left(-180\,Y^8-2640\,Y^6-70200\,Y^4-421200\,Y^2+45900\right)T$
$\mathbf{q}_2=3897234\,T^{10}+6\,Y^{10}+\left(1771470\,Y^2-17124210\right)T^8+270\,Y^8+(306180\,Y^4-1837080\,Y^2+13253220)T^6+13500\,Y^6+\left(24300\,Y^6+170100\,Y^4+5321700\,Y^2+19561500\right)T^4+78300\,Y^4+\left(810\,Y^8+20520\,Y^6+661500\,Y^4+3321000\,Y^2-984150\right)T^2-36450\,Y^2+12150$
$\mathbf{q}_1=-2125764\,T^{11}+\left(-1180980\,Y^2+12990780\right)T^9+(-262440\,Y^4+1574640\,Y^2-14959080)T^7+\left(-29160\,Y^6-320760\,Y^4-11284920\,Y^2-41319720\right)T^5+(-1620\,Y^8-58320\,Y^6-1927800\,Y^4-7938000\,Y^2+6374700)T^3+(-36\,Y^{10}-2340\,Y^8-83880\,Y^6-405000\,Y^4-429300\,Y^2-234900)T$, $\quad \mathbf{q}_0=531441\,T^{12}+Y^{12}+\left(354294\,Y^2-4369626\right)T^{10}+126\,Y^{10}+\left(98415\,Y^4-590490\,Y^2+7184295\right)T^8+3735\,Y^8+(14580\,Y^6+218700\,Y^4+8791740\,Y^2+34015140)T^6+15300\,Y^6+(1215\,Y^8+56700\,Y^6+1834650\,Y^4+4203900\,Y^2-3485025)T^4+143775\,Y^4+(54\,Y^{10}+4590\,Y^8+150300\,Y^6+639900\,Y^4+2782350\,Y^2+2020950)T^2+93150\,Y^2+2025$

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